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Observer-Dependence of Chaos Under Lorentz and Rindler Transformations

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The behavior of Lyapunov exponents λ and dynamical entropies h, whose positivity characterizes chaotic motion, under Lorentz and Rindler transformations is studied. Under Lorentz transformations, λ and h are changed, but their positivity is preserved for chaotic systems. Under Rindler transformations, λ and h are changed in such a way that systems, which are chaotic for an accelerated Rindler observer, can be nonchaotic for an inertial Minkowski observer. Therefore, the concept of chaos is observer-dependent.

KEY WORDS: chaos; Kolmogorov–Sinai entropy; Lyapunov exponent; Lorentz transformation; Rindler transformation.

A key concept to classify and characterize dynamical systems is their sensitivity to small changes of initial conditions. This sensitivity is expressed by the so-called Lyapunov exponents (LEs) λ_i that are defined as the statistical average

$$\lambda_i = \lim_{T\to\infty} \frac{1}{T} \int_0^T \lambda_i(t)\, dt \tag{1}$$

where

$$\lambda_i(t_o) = \lim_{\Delta t\to 0} \frac{1}{\Delta t} \log \left| \frac{\Delta x_i(t)}{\Delta x_i(t_o)} \right| \tag{2}$$

are local rates of change of the distance between different initial states (see Eckmann and Ruelle, 1985, for more details). Here, $\Delta x(t_o)$ is the distance between the initial states of the system, and $\Delta x(t)$ is the distance to which $\Delta x(t_o)$ has developed after time $\Delta t = t - t_o$ under the evolution of the system. The coordinates

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are indicated by $i = 1, \ldots, n$, where n is the dimension of the phase space in which the evolution (discrete or continuous in time t) of the state of the system is represented.

The LEs λ_i resulting from the temporal average of the ratio $\Delta x(t)/\Delta x(t_o)$ are known as dynamical invariants of the system. The invariance refers to transformations of the phase space coordinates (see, e.g., Eichhorn *et al.*, 2001). The evolution of the distance Δx can then be characterized by

$$\Delta x_t \approx \Delta x_{t_o} \exp(\lambda_i t) \tag{3}$$

Dynamical systems exhibit intrinsically unstable behavior if at least one LE of the system is positive although $\Sigma_i \lambda_i \leq 0$. In such a case, the distance between initial states is amplified exponentially. In recent decades, the term chaos (or deterministic chaos) has been coined and extensively used for this situation. Meanwhile there is a vast amount of literature in which many more details about chaotic dynamical systems are discussed from the perspectives of ergodic theory, vector fields, distributions, stochastic systems, etc. (see, e.g., Arnold, 1998; Cornfeld *et al.*, 1982; Eckmann and Ruelle, 1985; Guckenheimer and Holmes, 1983; Lasota and Mackey, 1994).

Since chaos is characterized by the positivity of at least one LE of the system, the significance of the concept of chaos depends essentially on the invariance of the LEs. In this paper we are interested in the behavior of the LEs under transformations beyond pure space transformations for dynamical systems. In particular, we study the behavior of LEs under two specific types of relativistic spacetime transformations between different observer frames: the Lorentz transformation and the Rindler transformation. For the observer frames, the natural choice is that of Minkowski and Rindler frames, respectively. We will demonstrate that a positive LE changes its value and can even become zero under these transformations.

It should be noted that chaotic behavior in general relativistic scenarios, in particular Bianchi type cosmological models, has been studied for quite a time (Barrow, 1982; Belinski *et al.*, 1982). Based on covariant definitions of "Lyapunov-like" exponents (Gurzadyan and Kocharyan, 1987; Szydlowski, 1993) and fractal dimensions (Cornish and Levin, 1997a,b), it was shown that chaos is an inherent feature of such cosmologies. Our purpose in this paper is different insofar as we study chaotic behavior in a given geometry, when different coordinate systems are employed by different observers, rather than the chaotic behavior of geometries themselves. To our knowledge the way in which an observer-dependence manifests itself for coordinate transformations with respect to a given geometry was not worked out before.

For an observer moving with speed v relative to an inertial observer, the Lorentz transformation (Einstein, 1905) provides a time dilation

$$\Delta t' = \frac{1}{\gamma}\Delta t \tag{4}$$

and a Lorentz contraction

$$\Delta x_1' = \gamma\,\Delta x_1 \tag{5}$$

$$\Delta x_i' = \Delta x_i \tag{6}$$

for $i = 2, 3$ with

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}} \tag{7}$$

The local rate of change of perturbations corresponding to the coordinate x_1 of a dynamical system at rest with respect to an inertial frame (t, x) is then obtained from

$$\lambda_{x_1}(t_o) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\ \log\left|\frac{\Delta x_1(t)}{\Delta x_1(t_o)}\right| \tag{8}$$

$$= \lim_{\Delta t' \to 0} \frac{1}{\gamma\,\Delta t'}\ \log\left|\frac{\Delta x_1(t')}{\Delta x_1(t_o')}\right|$$

$$- \lim_{\Delta t \to 0} \frac{1}{\Delta t}\ \log\left|\frac{\gamma}{\gamma}\right| \tag{9}$$

$$= \frac{1}{\gamma}\lambda'_{x_1}(t_o') \tag{10}$$

The LE of the system for $i = 1$ is then

$$\lambda_{x_1} = \lim_{T \to \infty} \frac{1}{T}\int_0^T \lambda_{x_1}(t)\,dt \tag{11}$$

$$= \lim_{T \to \infty} \frac{1}{T}\int_{\frac{-vx_1}{\gamma c^2}}^{\frac{1}{\gamma}(T - \frac{vx_1}{c^2})} \lambda'_{x1}(t')\,dt' \tag{12}$$

$$= \frac{1}{\gamma}\lambda'_{x_1} \tag{13}$$

with an appropriately defined λ'_{x_1} for the observer in the moving frame (for an example see the Appendix).

In addition to Eq. (13) ($i = 1$), the LEs for $i = 2, 3$ are

$$\lambda'_{x_i} = \gamma\lambda_{x_i} \tag{14}$$

For a canonical phase space with six coordinates, the Lorentz transformation for the remaining LEs (indexed by p) is obtained as ($i = 1, 2, 3$)

$$\lambda'_{p_i} = \gamma\lambda_{p_i} \tag{15}$$

These results show that the values of LEs are changed under the Lorentz transformation, but their positivity in the case of chaos is preserved. As a consequence, the concept of chaos is observer-independent under Lorentz transformations, although the degree of instability associated with chaotic motion changes. The changing factor γ is due to time dilation. As it is well known in the theory of dynamical systems, space transformations alone leave LEs invariant.

The second example to be discussed refers to transformations between an inertial Minkowski observer and an accelerated Rindler observer, for short Rindler transformations. A line element in the Minkowski coordinate frame (t, x) is given by

$$ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{16}$$

A line element in the Rindler coordinate frame (τ, ξ) is given by

$$ds^2 = -(1 + g\xi^1)^2 (d\tau)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 \tag{17}$$

where g is the proper acceleration of the local Rindler observer, resting at the origin of his frame, relative to the Minkowski frame.

The local Rindler transformation for the appropriate spacetime region is (Misner *et al.*, 1973)

$$t = (g^{-1} + \xi^1) \sinh(g\tau) \tag{18}$$

$$x^1 = (g^{-1} + \xi^1) \cosh(g\tau) \tag{19}$$

$$x^i = \xi^i \tag{20}$$

for $i = 2, 3$. Equivalently

$$\Delta t = \sinh(g\tau)\Delta\xi^1 + (1 + g\xi^1) \cosh(g\tau)\Delta\tau \tag{21}$$

$$\Delta x^1 = \cosh(g\tau)\Delta\xi^1 + (1 + g\xi^1) \sinh(g\tau)\Delta\tau \tag{22}$$

$$\Delta x^i = \Delta\xi^i \tag{23}$$

Let us first consider the case of a system resting in the Minkowski frame. Then we have $\Delta x^1 = 0$, hence from Eq. (22)

$$\Delta\xi^1 = -(1 + g\xi^1) \tanh(g\tau)\Delta\tau \tag{24}$$

Some calculation using Eqs. (21) and (19) then provides

$$\Delta t = (1 + g\xi^1)\frac{\Delta\tau}{\cosh(g\tau)} \tag{25}$$

$$= \frac{gx^1}{\cosh^2(g\tau)} \Delta_\tau \tag{26}$$

as the time dilation of a clock fixed in the Minkowski frame observed by a local Rindler observer. The coordinate time τ of the local Rindler system is also the proper time of the local Rindler observer resting at $\xi^i = 0$. The analogue of the Lorentz contraction along x^1 of a ruler resting in the Minkowski frame can be obtained as

$$\Delta\xi^1 = \frac{1}{\cosh(g\tau)}\Delta x^1 \tag{27}$$

The LEs of a dynamical system resting in the Minkowski frame are given by

$$\lambda_{x^i} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \lambda_{x^i}(t)\,dt \tag{28}$$

where

$$\lambda_{x^i}(t_o) = \lim_{\Delta t\to 0} \frac{1}{\Delta t} \log\left|\frac{\Delta x^i(t)}{\Delta x^i(t_o)}\right| \tag{29}$$

The corresponding LEs for a local Rindler observer are

$$\lambda'_{\xi^i} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \lambda'_{\xi_i}(\tau)\,d\tau \tag{30}$$

where

$$\lambda'_{\xi^i}(\tau_o) = \lim_{\Delta\tau\to 0} \frac{1}{\Delta\tau} \log\left|\frac{\Delta\xi^i(\tau)}{\Delta\xi^i(\tau_o)}\right| \tag{31}$$

Using the Rindler type time dilation and length contraction given by Eqs. (26) and (27), we get

$$\lambda'_{\xi^i}(\tau_o) = \lim_{\Delta t\to 0} \frac{gx^1}{\cosh^2(g\tau)}\frac{1}{\Delta t} \log\left|\frac{\Delta x^1(t)}{\Delta x^1(t_o)}\right| - \lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau} \log\left|\frac{\cosh(g\tau)}{\cosh(g\tau_o)}\right| \tag{32}$$

$$= \frac{gx^1}{\cosh^2(g\tau_o)}\lambda_{x^1}(t_o) - g\tanh(g\tau_o) \tag{33}$$

Therefore the LE for $i = 1$ transforms as

$$\lambda'_{\xi^1} = \lim_{T\to\infty} \frac{x^1\tanh(gT)}{Tx^1\tanh(gT)}\int_0^{x^1\tanh(gT)} \lambda_{x^1}(t)\,dt$$

$$- \lim_{T\to\infty}\frac{1}{T}\int_0^{gT} \tanh(g\tau)d(g\tau) \tag{34}$$

$$= \lambda_{x^1}\left(\lim_{T\to\infty}\frac{x^1}{T}\right) - g \tag{35}$$

where

$$\lambda_{x^1} = \frac{1}{x^1}\int_0^{x^1} \lambda_{x^1}(t)\,dt \tag{36}$$

is the corresponding LE for the Minkowski observer by Eqs. (19) and (30) (for an example see the Appendix).

As for the case of the Lorentz transformation, the picture for a six-dimensional canonical phase space can be completed by the transformation relations

$$\Delta p_{\xi^1} = \Delta p_{x^1} \cosh(g\tau) \tag{37}$$

$$\Delta \xi^i = \Delta x^i \tag{38}$$

$$\Delta p_{\xi^i} = \Delta p_{x^i} \tag{39}$$

with p_{ξ^i} and p_{x^i} as the momentum components conjugate to ξ^i and x^i and $i = 2$, 3. The corresponding LEs can be derived as

$$\lambda'_{p,\xi^1} = \lambda_{p,x^1} \left(\lim_{T\to\infty} \frac{x^1}{T} \right) + g \tag{40}$$

$$\lambda'_{\xi^i} = \lambda_{x^i} \left(\lim_{T\to\infty} \frac{x^1}{T} \right) \tag{41}$$

$$\lambda'_{p,\xi^i} = \lambda_{p,x^i} \left(\lim_{T\to\infty} \frac{x^1}{T} \right) \tag{42}$$

If the proper acceleration g of the Rindler observer vanishes, its status becomes that of a Minkowski observer and the LEs for the two frames are identical.

Applying Pesin's formula, the transformation properties of positive LEs for $g \neq 0$ can be expressed by the transformation of their sum, the Kolmogorov–Sinai (KS) entropy h, as

$$h' = h \left(\lim_{T\to\infty} \frac{x^1}{T} \right) + \alpha' g = \alpha' g \tag{43}$$

In ordinary units, the dimension of h is that of an inverse time. This can be taken into account by rewriting Eq. (43) as

$$h' = h \left(\lim_{T\to\infty} \frac{x^1}{T} \right) + \frac{\alpha g}{c} = \frac{\alpha g}{c} \tag{44}$$

where c is the velocity of light, so g/c is usually small. It follows that the value of the KS-entropy is proportional to g for the Rindler observer independently of its value in the Minkowski frame. Since the first term in Eq. (44) vanishes independently of h, any system in the Minkowski frame has $h' \propto g > 0$ in the Rindler frame. It is interesting to note that h' is formally proportional to the Hawking–Unruh temperature $\hat{T}$ since $g \propto \hat{T}$ (Hawking, 1974; Unruh, 1976).

By contrast to the situation considered so far, let us now consider a system resting at the origin of the Rindler frame ($\xi^i = 0$ and $\Delta\xi^i = 0$) and study the transformation properties of the LEs and the KS-entropy for this case. From Eq. (21), the time dilation of a clock fixed at the origin of the Rindler frame is

$$\Delta t = \cosh(g\tau)\Delta\tau \tag{45}$$

and, from Eq. (22), the length contraction of a ruler fixed in the Rindler frame along the ξ^1-axis is

$$\Delta x^1 = \frac{1}{\cosh(g\tau)}\Delta\xi^1 \tag{46}$$

From Eq. (29), we obtain

$$\lambda'_{\xi^1}(\tau_o) = \cosh(g\tau_o)\lambda_{x^1}(t_o) + g\ \tanh(g\tau_o) \tag{47}$$

which provides

$$\lambda'_{\xi^1} = \lim_{T\to\infty}\frac{1}{T}\int_0^{\sinh(gT)/g}\lambda_{x^1}(t)\,dt + \lim_{T\to\infty}\frac{1}{T}\int_0^{gT}\tanh(g\tau)\,d(g\tau_o) \tag{48}$$

$$= \lambda_{x^1}\left(\lim_{T\to\infty}\frac{\sinh(gT)}{gT}\right) + g \tag{49}$$

as the LE for $i = 1$ for the local Rindler observer. For the observer in the Minkowski frame this means

$$\lambda'_{x^1} = (\lambda_{\xi^1} - g)\left(\lim_{T\to\infty}\frac{gT}{\sinh(gT)}\right) \tag{50}$$

$$= \lambda'_{\xi^1}\left(\lim_{T\to\infty}\frac{gT}{\sinh(gT)}\right) \tag{51}$$

The LEs corresponding to the other phase space coordinates are transformed as ($i = 2, 3$)

$$\lambda'_{p,x^1} = (\lambda_{p,\xi^1} + g)\left(\lim_{T\to\infty}\frac{gT}{\sinh(gT)}\right) = \lambda'_{p,\xi^1}\left(\lim_{T\to\infty}\frac{gT}{\sinh(gT)}\right) \tag{52}$$

$$\lambda_{x^i} = \lambda'_{\xi^i}\left(\lim_{T\to\infty}\frac{gT}{\sinh(gT)}\right) \tag{53}$$

$$\lambda_{p,x^i} = \lambda'_{p,\xi^i}\left(\lim_{T\to\infty}\frac{gT}{\sinh(gT)}\right) \tag{54}$$

The KS-entropy h for the Minkowski observer is

$$h = h'\left(\lim_{T\to\infty}\frac{gT}{\sinh(gT)}\right) \tag{55}$$

where h' is the KS-entropy for the Rindler observer. Equation (55) provides $h = h'$ for $g = 0$, when the local Rindler observer effectively becomes a Minkowski observer.

In the general case of $g \neq 0$, h vanishes for any finite nonnegative value of h'. In other words, a chaotic system ($h' > 0$) in the local Rindler frame will be nonchaotic ($h = 0$) in the Minkowski frame. Conversely, h' tends to infinity for any finite value of h. This implies that the degree of instability of a chaotic system in the Minkowski frame diverges if it is transformed to the Rindler frame. As will be shown elsewhere, the situation is similar for the transformation between Novikov and Schwarzschild observers.

In summary, we have shown that the concept of chaos as characterized by positive LEs or their sum, the KS-entropy, is not invariant under the Rindler transformation. In particular, any nonchaotic system in the Minkowski frame is chaotic in the local Rindler frame, and any chaotic system in the local Rindler frame is nonchaotic in the Minkowski frame. Under the Lorentz transformation, the concept of chaos itself is invariant but the degree of instability related to chaotic motion is not.

The main origin of the observer-dependence of chaos under Rindler transformations as in Eq. (55) and in the first term of Eqs. (43) and (44) is the exponential time dilation. (The general significance of nonlinearities in spacetime transformations of chaotic systems will be discussed elsewhere.) The second term in Eqs. (43) and (44) adds another increment to this observer-dependence, which results from length contraction. This increment is formally proportional to the Hawking–Unruh temperature, which is induced by Hawking–Unruh thermal noise related to quantum fluctuations. The observer-dependence of chaos, however, is a purely classical effect.

Other interesting features in this context are the transformations of Minkowski vacuum states into Rindler particles (see Clifton and Halvorson, 2001, for a most recent account) and of thermal equilibrium states in Minkowksi spacetime into degenerate temperature states in Rindler spacetime (Zhao *et al.*, 1996).

APPENDIX: A SIMPLE EXAMPLE

Consider an automorphism Φ on a torus orthogonal to the x_1-axis given by

$$\Phi = (ax_2 + bx_3, cx_2 + dx_3) \pmod 1 \tag{A1}$$

with $ad - bc = 1$. The eigenvalues of the coefficient matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A2}$$

are μ_i for $i = 1, 2$. It is known that Φ is chaotic if the two roots of M are incommensurable. In this case, the KS-entropy is given by

$$h = \log |\mu_m| \tag{A3}$$

where μ_m is that value of μ_i whose modulus is greater than 1.

A special case of such an automorphism is Arnold's cat map with $a = b = c = 1$ and $d = 2$. In this case, $h = \log(3 + \sqrt{5})/2$.

Lorentz Transformation

Now consider the torus resting in the inertial frame. For an observer resting with respect to the torus, let us assume the time interval $\Delta t = 1$ for the automorphism. The corresponding time for the moving observer then is

$$\Delta t' = \Delta t \frac{1}{\gamma} = \frac{1}{\gamma} \tag{A4}$$

The time interval of the automorphism during a unit of time t' is $1/\Delta t'$ for the moving observer. Hence the coefficient matrix is

$$M' = M^{\frac{1}{\Delta t'}} \tag{A5}$$

with eigenvalues

$$\mu_i' = \mu_i^{\frac{1}{\Delta t'}} \tag{A6}$$

As a result, the KS-entropy for the moving observer is

$$h' = \log |\mu_m|^{\frac{1}{\Delta t'}} = \gamma h \tag{A7}$$

Rindler Transformation

Let us consider the same torus, again resting in the Minkowski frame. For a Minkowski observer resting close to the torus, let us assume the time interval $\Delta t = 1$ for the automorphism. Since $\Delta x^1 = 0$, we have

$$\Delta\xi^1 = -(1 + g\xi^1)\tanh(g\tau)\Delta\tau \tag{A8}$$

With

$$\Delta t = (1 + g\xi^1)\frac{\Delta\tau}{\cosh(g\tau)} \tag{A9}$$

and

$$1 + g\xi^1 = \frac{gx^1}{\cosh(g\tau)} \tag{A10}$$

due to Eq. (19), we obtain

$$\Delta t = \frac{gx^1}{\cosh^2(g\tau)}\Delta\tau \tag{A11}$$

as the "time dilation" of a clock fixed in the Minkowski frame for the local Rindler observer. For him, the time interval of the automorphism during a unit of time τ is

$$\frac{1}{\Delta\tau} = \frac{gx^1}{\cosh^2(g\tau)} \tag{A12}$$

Hence the coefficient matrix is

$$M' = M^{\frac{1}{\Delta\tau}} \tag{A13}$$

with eigenvalues

$$\mu_i' = \mu_i^{\frac{1}{\Delta\tau}} \tag{A14}$$

The KS-entropy for the local Rindler observer is

$$h' = \lim_{T\to\infty} \frac{1}{T} \int_0^T h'(\tau)\, d\tau \tag{A15}$$

where

$$h'(\tau) = \log|\mu_m|^{\frac{1}{\Delta\tau}} = \frac{g\xi^1}{\cosh^2(g\tau)} h \tag{A16}$$

Hence,

$$h' = \lim_{T\to\infty} \frac{x^1 h}{T} \int_0^{gT} \frac{1}{\cosh^2(g\tau)}\, d(g\tau) \tag{A17}$$

$$= h\left(\lim_{T\to\infty} \frac{x^1}{T}\right) \tag{A18}$$

As a result, the KS-entropy vanishes for the local Rindler observer if it is finite for the Minkowski observer.

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